Answer to a question of Alon and Lubetzky about the ultimate categorical independence ratio

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Abstract

Brown, Nowakowski and Rall defined the ultimate categorical independence ratio of a graph G as $A(G) = \lim_{k\to\infty} i(G^{\times k})$, where $i(G) = \frac{\alpha(G)}{|V(G)|}$ denotes the independence ratio of a graph G, and $G^{\times k}$ is the kth categorical power of G. Let $a(G) = \max\{\frac{|U|}{|U|+|N_G(U)|} : U \text{ is an independent set of } G\}$, where $N_G(U)$ is the neighborhood of U in G. In this paper we answer a question of Alon and Lubetzky, namely we prove that if $a(G) \le \frac{1}{2}$ then $A(G) = a(G)$, and if $a(G) > \frac{1}{2}$ then $A(G) = 1$. We also discuss some other open problems related to $A(G)$ which are immediately settled by this result.

1 Introduction

The *independence ratio* of a graph G is defined as $i(G) = \frac{\alpha(G)}{|V(G)|}$, that is, as the ratio of the independence number and the number of vertices. For two graphs G and H, their *categorical product* (also called as direct or tensor product) $G\times H$ is defined on the vertex set $V(G\times H) = V(G)\times V(H)$ with edge set $E(G\times H) = \{\{(x_1, y_1), (x_2, y_2)\} : \{x_1, x_2\} \in E(G) \text{ and } \{y_1, y_2\} \in E(H)\}$. The kth categorical power $G^{\times k}$ is the k-fold categorical product of G. The *ultimate categorical independence ratio* of a graph G is defined as

$$A(G) = \lim_{k\to\infty} i(G^{\times k}).$$

This parameter was introduced by Brown, Nowakowski and Rall in [2] where they proved that for any independent set U of G the inequality $A(G) \ge \frac{|U|}{|U|+|N_G(U)|}$ holds, where $N_G(U)$ denotes the neighborhood of U in G. Furthermore, they showed that $A(G) > \frac{1}{2}$ implies $A(G) = 1$.

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Motivated by these results, Alon and Lubetzky [1] defined the parameters $a(G)$ and $a^*(G)$ as follows

$$a(G) = \max_{U \text{ is indep. set of } G} \frac{|U|}{|U| + |N_G(U)|} \qquad \text{and} \qquad a^*(G) = \begin{cases} a(G) & \text{if } a(G) \le \frac{1}{2} \\ 1 & \text{if } a(G) > \frac{1}{2} \end{cases},$$

and they proposed the following two questions.

Question 1 ([1]). Does every graph G satisfy $A(G) = a^*(G)$? Or, equivalently, does every graph G satisfy $a^*(G^{\times 2}) = a^*(G)$?

Question 2 ([1]). Does the inequality $i(G \times H) \le \max\{a^*(G), a^*(H)\}$ hold for every two graphs G and H?

The above results from [2] give us the inequality $A(G) \ge a^*(G)$. One can easily see the equivalence between the two forms of Question 1, moreover it is not hard to show that an affirmative answer to Question 1 would imply the same for Question 2 (see [1]).

Following [2] a graph G is called self-universal if $A(G) = i(G)$. As a consequence, the equality $A(G) = a^*(G)$ in Question 1 is also satisfied for these graphs according to the chain inequality $i(G) \le a(G) \le a^*(G) \le A(G)$. Regular bipartite graphs, cliques and Cayley graphs of Abelian groups belong to this class [2]. In [4] the author proved that a complete multipartite graph is self-universal, except for the case when $a(G) > \frac{1}{2}$, therefore the equality $A(G) = a^*(G)$ is also verified for this class of graphs. (In the latter case $A(G) = a^*(G) = 1$.) In [1] it is shown that the graphs which are disjoint union of cycles and complete graphs satisfy the inequality in Question 2.

In this paper we answer Question 1 affirmatively. Thereby a positive answer also for Question 2 is obtained. Moreover it solves some other open problems related to $A(G)$. In the proofs we exploit an idea of Zhu [3] that he used on the way when proving the fractional version of Hedetniemi's conjecture. In Section 2 this tool is presented. Then, in Section 3, first we prove the inequality

$$i(G \times H) \le \max\{a(G), a(H)\}, \quad \text{for every two graphs } G \text{ and } H,$$

and give a positive answer to Question 2 (using $a(G) \le a^*(G)$). Afterwards we prove that

$$a(G \times H) \le \max\{a(G), a(H)\}, \quad \text{provided that } a(G) \le \frac{1}{2} \text{ or } a(H) \le \frac{1}{2},$$

and from this result we conclude the affirmative answer to Question 1. (If $a(G) > \frac{1}{2}$ then $a^*(G^{\times 2}) = a^*(G) = 1$. Otherwise applying the above result for $G = H$ we get $a(G^{\times 2}) \le a(G)$, while the reverse inequality clearly holds for every G. Thus we have $a^*(G^{\times 2}) = a^*(G)$ for every graph G.) Finally, in Section 4, we discuss further open problems which are solved by our result. For instance, we get a proof for the conjecture of Brown, Nowakowski and Rall, stating that $A(G \cup H) = \max\{A(G), A(H)\}$, where $G \cup H$ is the disjoint union of G and H.

2 Zhu's lemma

Recently Zhu [3] proved the fractional version of Hedetniemi's conjecture, that is, he showed that for every graph G and H we have $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$, where $\chi_f(G)$ denotes the fractional chromatic number of the graph G. During the proof he showed the following result on the independent sets of categorical product of graphs. This will be the key idea also in our case.

Let U be an independent set of $G \times H$. Zhu considered the partition U into $U = A \uplus B$, where

$$\begin{aligned} A &= \{(x,y) \in U \,:\, \nexists (x',y) \in U \text{ s.t. } \{x,x'\} \in E(G)\}, \\ B &= \{(x,y) \in U \,:\, \exists (x',y) \in U \text{ s.t. } \{x,x'\} \in E(G)\}. \end{aligned} \tag{1}$$

In the sequel, we keep using the following notations for any $Z \subseteq V(G \times H)$.
For any $y \in V(H)$, let

$$Z(y) = \{x \in V(G) \,:\, (x,y) \in Z\}.$$

Similarly, for any $x \in V(G)$, let

$$Z(x) = \{y \in V(H) \,:\, (x,y) \in Z\}.$$

And, let

$$N^G(Z) = \{(x,y) \in V(G \times H) \,:\, x \in N_G(Z(y))\}.$$

In words, $N^G(Z)$ means that we decompose Z into sections corresponding to the elements of $V(H)$, and in each section we pick those points which are neighbors of the elements of $Z(y)$ in the graph G. Similarly, let

$$N^H(Z) = \{(x,y) \in V(G \times H) \,:\, y \in N_H(Z(x))\}.$$

Keep in mind, that $Z(y) \subseteq V(G)$ and $Z(x) \subseteq V(H)$, while $N^G(Z), N^H(Z) \subseteq V(G \times H)$.

Lemma 1 ([3]). *The following holds:*
(1) For every $y \in V(H)$, $A(y)$ is an independent set of G. For every $x \in V(G)$, $B(x)$ is an independent set of H.
(2) A, B, $N^G(A)$ and $N^H(B)$ are pairwise disjoint subsets of $V(G \times H)$.

For the sake of completeness we prove this lemma.

Proof. $A(y)$ is independent for every $y \in V(H)$ by definition. If for any $x \in V(G)$ the set $B(x)$ is not independent in H, that is $\exists y, y' \in B(x)$, $\{y, y'\} \in E(H)$, then from $(x, y') \in B$ we get that $\exists (x', y') \in U$, $\{x, x'\} \in E(G)$. This is a contradiction, because $(x, y) \in B$ and $(x', y') \in U$ were two adjacent elements of the independent set U.

Now we show the second part of the lemma. By definition $A \cap B = \emptyset$. The first part of the lemma implies that the pair $(A, N^G(A))$ is also disjoint, as well as the pair $(B, N^H(B))$.

If $(x,y) \in A \cap N^H(B)$ then (by the definition of $N^H(B)$) $\exists (x,y') \in B$, $\{y,y'\} \in E(H)$, and so (by the definition of B) $\exists (x',y') \in U$, $\{x,x'\} \in E(G)$, which is a contradiction: $(x,y) \in A$ and $(x',y') \in U$ are adjacent vertices in the independent set U. Similarly, if $(x,y) \in N^G(A) \cap N^H(B)$ then (by the definition of $N^G(A)$) $\exists (x',y) \in A \subseteq U$, $\{x,x'\} \in E(G)$ while (by the definition of $N^H(B)$) $\exists (x,y') \in B \subseteq U$, $\{y,y'\} \in E(H)$, which contradicts to the independence of U. Finally, $(x,y) \in B \cap N^G(A)$ implies that $\exists (x',y) \in A$, $\{x,x'\} \in E(G)$ (by the definition of $N^G(A)$), which is in contradiction with the definition of A: there should not be an $(x,y) \in B \subseteq U$ satisfying $\{x,x'\} \in E(G)$. $\square$

3 Proofs

In this section we prove the statements mentioned in the Introduction. In Subsection 3.1 we give an upper bound for $i(G \times H)$ in terms of $a(G)$ and $a(H)$. In Subsection 3.2 we prove that the same upper bound holds also for $a(G \times H)$ provided that $a(G) \le \frac{1}{2}$ or $a(H) \le \frac{1}{2}$. Thereby we obtain that $A(G) = a^*(G)$ for every graph G.

3.1 Upper bound for $i(G \times H)$

As a simple consequence of Zhu's result the following inequality is obtained.

Theorem 2. *For every two graphs G and H we have*

$$i(G \times H) \le \max\{a(G), a(H)\}.$$

Proof. Let U be a maximum-size independent set of $G \times H$, then we have

$$i(G \times H) = \frac{\alpha(G \times H)}{|V(G \times H)|} = \frac{|U|}{|V(G \times H)|}. \tag{2}$$

We partition U into $U = A \uplus B$ according to (1). We also use the notations $A(y)$ for every $y \in V(H)$, $B(x)$ for every $x \in V(G)$, and $N^G(A)$, $N^H(B)$ defined in the previous section.
It is clear that $|U| = |A| + |B|$. From the second part of Lemma 1 we have that $|A| + |B| + |N^G(A)| + |N^H(B)| \le |V(G \times H)|$. Observe that $|N^G(A)| = \sum_{y \in V(H)} |N_G(A(y))|$ and $|N^H(B)| = \sum_{x \in V(G)} |N_H(B(x))|$. Hence we get

$$\begin{aligned} \frac{|U|}{|V(G \times H)|} \le \frac{|A| + |B|}{|A| + |B| + |N^G(A)| + |N^H(B)|} = \\ = \frac{\sum_{y \in V(H)} |A(y)| + \sum_{x \in V(G)} |B(x)|}{\sum_{y \in V(H)}(|A(y)| + |N_G(A(y))|) + \sum_{x \in V(G)}(|B(x)| + |N_G(B(x))|)}. \end{aligned} \tag{3}$$

From the first part of Lemma 1 and by the definition of $a(G)$ and $a(H)$ we have $\frac{|A(y)|}{|A(y)|+|N_G(A(y))|} \le a(G)$ for every $y \in V(H)$, and $\frac{|B(x)|}{|B(x)|+|N_H(B(x))|} \le a(H)$ for every $x \in V(G)$, respectively.

Using the fact that if $\frac{t_1}{s_1} \le r$ and $\frac{t_2}{s_2} \le r$ then $\frac{t_1+t_2}{s_1+s_2} \le r$, this yields

$$\frac{\sum_{y\in V(H)} |A(y)| + \sum_{x\in V(G)} |B(x)|}{\sum_{y\in V(H)}(|A(y)| + |N_G(A(y))|) + \sum_{x\in V(G)}(|B(x)| + |N_H(B(x))|)} \le \max\{a(G), a(H)\}. \qquad (4)$$

The inequalities (2), (3) and (4) together give us the stated inequality,

$$i(G \times H) \le \max\{a(G), a(H)\}.$$

□

As we stated in the Introduction, from Theorem 2 it follows that the answer to Question 2 is positive.

3.2 Answer to Question 1

In this subsection we answer Question 1 affirmatively. To show that $a^*(G^{\times 2}) = a^*(G)$ holds for every graph G it is enough to prove that $a(G^{\times 2}) \le a(G)$ for every graph G with $a(G) \le \frac{1}{2}$. Because if $a(G) > \frac{1}{2}$ then $a^*(G^{\times 2}) = a^*(G) = 1$, in addition every G satisfies $a(G^{\times 2}) \ge a(G)$. The condition $a(G) \le \frac{1}{2}$ is necessary, since otherwise $A(G) = 1$ therefore $i(G^{\times k})$ and $a(G^{\times k})$ as well can be arbitrary close to 1 for sufficiently large k. A bit more general, we prove the following theorem.

Theorem 3. *If $a(G) \le \frac{1}{2}$ or $a(H) \le \frac{1}{2}$ then*

$$a(G \times H) \le \max\{a(G), a(H)\}.$$

Proof. We will show that for every independent set U of $G \times H$ we have

$$\frac{|U|}{|U| + |N_{G\times H}(U)|} \le \max\{a(G), a(H)\}.$$

First, let $\hat{A}$, $\hat{B}$ and C be the following subsets of U.

$$\hat{A} = \{(x, y) \in U \ : \ \nexists (x', y) \in U \text{ s.t. } \{x, x'\} \in E(G), \text{ but } \exists (x, y') \in U \text{ s.t. } \{y, y'\} \in E(H)\},$$
$$\hat{B} = \{(x, y) \in U \ : \ \nexists (x, y') \in U \text{ s.t. } \{y, y'\} \in E(H), \text{ but } \exists (x', y) \in U \text{ s.t. } \{x, x'\} \in E(G)\},$$
$$C = \{(x, y) \in U \ : \ \nexists (x', y) \in U \text{ s.t. } \{x, x'\} \in E(G), \text{ and } \nexists (x, y') \in U \text{ s.t. } \{y, y'\} \in E(H)\}.$$

It is clear that $\hat{A}$, $\hat{B}$ and C are pairwise disjoint. In addition, there is no $(x, y) \in U$ for which $\exists (x', y), (x, y')$ in U such that $\{x, x'\} \in E(G)$ and $\{y, y'\} \in E(H)$, because $\{(x', y), (x, y')\} \in E(G \times H)$ and U is an independent set. Hence **U is partitioned into $\mathbf{U} = \hat{\mathbf{A}} \uplus \hat{\mathbf{B}} \uplus \mathbf{C}$**. (The connection with the partition of Zhu defined in (1) is clearly the following, $A = \hat{A} \uplus C$ and $B = \hat{B}$.)

Observe that the definition of $a(G)$ can be rewritten as follows

$$\min\left\{\frac{|N_G(U)|}{|U|} \ : \ \text{U is independent in } G\right\} = \frac{1 - a(G)}{a(G)}.$$

Set $b(G) = \frac{1-a(G)}{a(G)}$. It it enough to prove that $|\mathbf{N_{G\times H}(U)}| \geq \mathbf{min\{b(G), b(H)\}|U|}$. We shall give a lower bound for $|N_{G\times H}(U)|$ in two steps.

In the **first step** we consider the elements of $\hat{A}$ and C for every $y \in V(H)$. By definition $(\hat{A}\cup C)(y)$ is independent in G for every $y \in V(H)$, therefore $|N_G((\hat{A}\cup C)(y))| \geq b(G)|(\hat{A}\cup C)(y)|$. We partition $N^G(\hat{A}\cup C)$ into two parts, let

$$N_1 = N^G(\hat{A}\cup C)\cap N_{G\times H}(U) \quad \text{and} \quad M = N^G(\hat{A}\cup C)\setminus N_{G\times H}(U).$$

(It is easy to see that $N^G(\hat{A}) \subseteq N_{G\times H}(U)$. However $N^G(C) \subseteq N_{G\times H}(U)$ is not necessarily true, that is why we make this partition.) Thus for $N_1 \subseteq N_{G\times H}(U)$ we have

$$|N_1| \geq b(G)\big(|\hat{A}| + |C|\big) - |M|. \tag{5}$$

In the **second step** we consider the elements of $\hat{B}$ and M for every $x \in V(G)$. By the definition of $\hat{A}$ and C, $\hat{B}(x)$ and $M(x)$ are disjoint. Indeed, if $(x,y) \in M \subseteq N^G(\hat{A}\cup C)$ then $\exists (x',y) \in \hat{A}\cup C, \{x,x'\} \in E(G)$ and so (x,y) cannot be in $\hat{B} \subseteq U$.

We claim that $(\hat{B}\cup M)(x)$ is independent in $V(H)$. Clearly, $\hat{B}(x)$ is independent. Furthermore, if $y, y' \in M(x), \{y,y'\} \in E(H)$ then from $(x,y) \in M$ we get that $\exists (x',y) \in \hat{A}\cup C, \{x,x'\} \in E(G)$, hence $(x,y') \in M$ is a neighbor of $(x',y) \in U$ which contradicts that $M \cap N_{G\times H}(U) = \emptyset$. Similarly if $y \in \hat{B}(x), y' \in M(x), \{y,y'\} \in E(H)$ then from $(x,y) \in \hat{B}$ it follows that $\exists (x',y) \in U, \{x,x'\} \in E(G)$, but again, as $(x,y') \in M$ is a neighbor of $(x',y) \in U$ it is in a contradiction with the definition of M. Therefore $|N_H((\hat{B}\cup M)(x))| \geq b(G)|(\hat{B}\cup M)(x)|$. Let

$$N_2 = N^H(\hat{B}\cup M).$$

Considering the sum for all $x \in V(G)$ we obtain

$$|N_2| \geq b(H)\big(|\hat{B}| + |M|\big). \tag{6}$$

We show that $N_2 \subseteq N_{G\times H}(U)$. On the one hand, if $y \in \hat{B}(x)$ and y' is a neighbor of y in H, and so $(x,y') \in N^H(\hat{B})$ then by the definition of $\hat{B}$, $\exists (x',y) \in U, \{x,x'\} \in E(G)$, hence (x,y') is a neighbor of $(x',y) \in U$, that is, $(x,y') \in N_{G\times H}(U)$. On the other hand, if $y \in M(x)$ and y' is a neighbor of y in H, and so $(x,y') \in N^H(M)$ then by the definition of M, $\exists (x',y) \in \hat{A}\cup C, \{x,x'\} \in E(G)$, therefore $\{(x',y),(x,y')\} \in E(G\times H)$, thus $(x,y') \in N_{G\times H}(U)$.

Next we prove that the neighborhood sets gotten in the two steps, $\mathbf{N_1}$ **and** $\mathbf{N_2}$ **are disjoint**. Suppose indirectly, that $(x,y) \in N_1 \cap N_2$. Then $(x,y) \in N_1$ implies that $\exists (x',y) \in \hat{A}\cup C, \{x,x'\} \in E(G)$. While from $(x,y) \in N_2$ we get that $\exists (x,y') \in \hat{B}$ or $\exists (x,y') \in M$ satisfying $\{y,y'\} \in E(H)$. It is a contradiction since (x',y) and (x,y') are adjacent in $G\times H$, but no edge can go between $\hat{A}\cup C$ and $\hat{B}\cup M$ by the independence of U and the definition of M. As $N_1, N_2 \subseteq N_{G\times H}(U)$ this yields

$$|N_{G\times H}(U)| \geq |N_1| + |N_2|. \tag{7}$$

From (5), (6) and (7) we obtain that

$$|N_{G\times H}(U)| \geq |N_1| + |N_2| \geq \Big(b(G)\big(|\hat{A}| + |C|\big) - |M|\Big) + \Big(b(H)\big(|\hat{B}| + |M|\big)\Big).$$

If $\mathbf{a(H)} \leq \mathbf{\frac{1}{2}}$, that is $b(H) \geq 1$, then

$$\Big(b(G)\big(|\hat{A}| + |C|\big) - |M|\Big) + \Big(b(H)\big(|\hat{B}| + |M|\big)\Big) \geq$$
$$\geq \min\{b(G), b(H)\}\Big(|\hat{A}| + |\hat{B}| + |C|\Big) + \big(b(H) - 1\big)|M| \geq \min\{b(G), b(H)\}|U|.$$

Combining the latter two inequalities we obtain $|N_{G\times H}(U)| \geq \min\{b(G), b(H)\}|U|$, as desired.

If $\mathbf{a(G)} \leq \mathbf{\frac{1}{2}}$ (and $a(H) > \frac{1}{2}$) we can change the role of G and H to get the same lower bound for $|N_{G\times H}(U)|$, or we can argue as follows. We distinguish two cases. First, suppose $|\hat{A}|+|C|-\frac{|M|}{b(G)} \geq 0$. By using $b(G) \geq 1$ this gives

$$\Big(b(G)\big(|\hat{A}| + |C|\big) - |M|\Big) + \Big(b(H)\big(|\hat{B}| + |M|\big)\Big) = b(G)\left(|\hat{A}| + |C| - \frac{|M|}{b(G)}\right) + b(H)\big(|\hat{B}| + |M|\big) \geq$$
$$\geq \min\{b(G), b(H)\}\left(|\hat{A}| + |\hat{B}| + |C| + |M|\big(1 - \frac{1}{b(G)}\big)\right) \geq \min\{b(G), b(H)\}|U|,$$

finishing the inequality chain. While from $|\hat{A}|+|C|-\frac{|M|}{b(G)} < 0$ and $b(G) \geq 1$ it follows $|\hat{A}|+|C| < |M|$, hence we have

$$|N_{G\times H}(U)| \geq |N_2| \geq b(H)\big(|\hat{B}| + |M|\big) \geq \min\{b(G), b(H)\}|U|.$$

Consequently, $|N_{G\times H}(U)| \geq \min\left\{\frac{1-a(G)}{a(G)}, \frac{1-a(H)}{a(H)}\right\}|U|$ in both cases, that is $\frac{|U|}{|U|+|N_{G\times H}(U)|} \leq \max\{a(G), a(H)\}$, this completes the proof. $\square$

We mentioned in the Introduction that the two forms of Question 1 are equivalent. Hence from the equality $a^*(G^{\times 2}) = a^*(G)$ for every graph G we obtain the following corollary. (Indeed, suppose on the contrary that G is a graph with $a^*(G) < A(G)$ then $\exists k$ such that $a^*(G) < i(G^{\times k}) \leq a^*(G^{\times k})$, and as the sequence $\{a^*(G^{\times \ell})\}_{\ell=1}^{\infty}$ is monotone increasing, it follows that $\exists m$ for which $a^*(G^{\times m}) < a^*(G^{\times 2m})$, giving a contradiction.)

Corollary 4. *For every graph G we have $A(G) = a^*(G)$.*

4 Further consequences

Brown, Nowakowski and Rall in [2] asked whether $A(G \cup H) = \max\{A(G), A(H)\}$, where $G \cup H$ denotes the disjoint union of G and H. From Corollary 4 we immediately receive this equality since the analogue statement, $a^*(G \cup H) = \max\{a^*(G), a^*(H)\}$ is straightforward. In [1] it is shown that $A(G \cup H) = A(G \times H)$, therefore we have

$$A(G \cup H) = A(G \times H) = \max\{A(G), A(H)\}, \text{ for every graph } G \text{ and } H.$$

The authors of [2] also addressed the question whether $A(G)$ is computable, and if so what is its complexity. They showed that if G is bipartite then $A(G) = \frac{1}{2}$ if G has a perfect matching, and $A(G) = 1$ otherwise. Hence for bipartite graphs $A(G)$ can be determined in polynomial time. Moreover, it is proven in [1] that $a(G) \leq \frac{1}{2}$ if and only if G contains a fractional perfect matching. Therefore given an input graph G, determining whether $A(G) = 1$ or $A(G) \leq \frac{1}{2}$ can be done in polynomial time. They also mentioned that deciding whether $a(G) > t$ for a given graph G and a given value t, is NP-complete. From Corollary 4 we can conclude that $A(G)$ can be calculated, and the problem of deciding whether $A(G) > t$ is NP-complete too.

Although any rational number in $(0, \frac{1}{2}] \cup \{1\}$ is the ultimate categorical independence ratio for some graph G, as it is showed [2]. Here we remark that we obtained that $A(G)$ cannot be irrational, solving another problem mentioned in [2].

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